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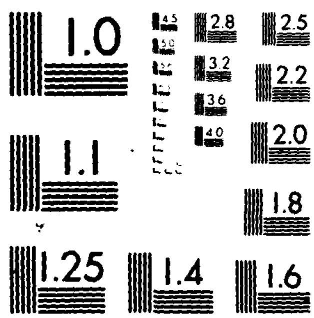

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Assessing Risks Through the Determination of
Rare Event Probabilities

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Assessing Risks Through the Determination of Rare Event Probabilities

by

Allan R. Sampson and Robert L. Smith

Abstract

We consider the problem in risk assessment of evaluating the probability of occurrence of rare, but potentially catastrophic, events. The lack of historical data due to the sheer novelty of the event makes conventional statistical approaches inappropriate. The problem is compounded by the complex multivariate dependencies that may exist across potential event sites. In order to evaluate the likelihood of one or more such catastrophic events occuring, we provide an information theoretic model for merging a decision maker's opinion with expert judgment. Also provided is a methodology for the reconciling of conflicting expert judgments. This merging approach is invariant to the decision maker's viewpoint in the limiting case of exceptionally rare events. These methods are applied to case studies in likelihood assessment of Liquid Natural Gas tanker spills and seismic induced light water nuclear reactor meltdowns.

KEY WORDS: risk assessment, information theory, merging opinions, rare events, Fréchét bounds, LNG tanker spills, nuclear reactor safety.

Assessing Risks Through the Determination of Rare Event Probabilities

by

Allan R. Sampson* and Robert L. Smith

1. INTRODUCTION

In an increasingly complex and technological world, methodologies for dealing with risk assessment have taken on new importance. The scale and power of technology have created the potentiality for the occurrence of truly catastrophic events. These include, but are not limited to, LNG (liquid natural gas) tanker explosions, serious nuclear reactor accidents, recombinant DNA accidents, and nuclear weapons accidents. Nevertheless, the more remote the likelihood of such an occurrence, the less apprehensive we are, until some threshold probability P_T is reached below which we become indifferent. This threshold value of course depends heavily on the seriousness of the consequences were the event to occur. Risk assessment therefore can be viewed as a two-fold process comprising i) the assessment of the likelihood of an undesireable event occurring (risk determination, see Rowe (1977)), and ii) the valuation of the consequences to the risk taker were that event to occur (risk evaluation, see Rowe). Our focus in this paper is on the first problem of risk determination.

Risk determination is a particularly difficult task because of the sheer novelty of the event of concern. Being catastrophic, it is almost certainly a rare event, and moreover may never have historically occurred. This lack of historical statistical data makes it

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difficult, if not impossible, to use conventional data analysis to assess the likelihood of the event's occurrence. We are in a situation of formulating a probability model, where the model must be based on limited structural information. Concerning these concepts in assessing nuclear reactor safety, Lewis (1980) writes that

> "a reactor accident with public consequences is demonstrably an extremely improbable event for which there is no actuarial record. When any event, such as a putative reactor accident, has a very low probability, but potentially severe consequences, it means that safety analysis must be based on theoretical calculations rather than on experience and this leads to special problems."

The decision maker (who must ultimately make the determination as to how "safe" the relevant technology or system is) is prototypically confronted with probability judgments from one or more experts, often in conflict with his own personal assessment as well as with each other. The term "experts" is used generically to denote individuals, commissions, study groups, etc. The decision maker's task is to reconcile the conflicts and merge the opinions in a responsible and consistent fashion. The probability assessment (or range of assessments) thus synthesized may be compared with the threshold probability value and a decision may be reached as to whether or not action is warranted.

We propose in this paper a methodology that will provide for a consistent merging of the decision maker's opinions with expert judgments as to the event's likelihood. We show that for rare events, the model provides for 1st order agreement as to the event's likelihood no matter what the initial views of the decision maker. Further, the model leads to a sensible procedure for reconciling, merging, and updating differing expert judgments.

Consider now two case studies in risk assessment. The first case will later illustrate the merging of a decision maker's viewpoint with expert judgment, and the second will illustrate the model's procedure for reconciling differing expert opinions.

Case Study 1 (LNG Tanker Movement)

The possibility of a large spill of LNG (Liquid Natural Gas) in a harbor due to a tanker collision is a potential hazard of considerable concern. It is widely held that such an occurrence could result in a catastrophic conflagration whose consequences are of the same order of magnitude in lives and property as a major accident at a nuclear power plant. It is therefore essential to obtain estimates of the likelihood of such an event occuring.

In 1974, the Federal Power Commission requested a report by independent consultants as to the risk posed by proposed LNG Tanker movement in the New York harbor. Table 1 (taken from Fairley (1977)) represents a summary of the central findings of that report. The

Table 1.* Summary of Commission Tanker Report

Factor	Symbol	Commission report estimate
Number of trips	A	1245**
Historical probability of a tankship or tank barge accident	B	8/10,000
LNG navigation reduction factor	C	1/5
Right tail area of damage distribution	D	1/100
Probability that damage occurs in tank area	E	2/3
Probability of spillage for a tankship or tank barge	F	1
Reduction factor engineering design of an LNG vessel as compared to a tankship or tank vessel	G	1/5
Probability of an undesirable event given the foregoing	H	1
Probability of an undesirable LNG event in 10 years (product of factors A through H)		1/3745
Single trip probability (product of factors B through H)		1/4,664,179

*Adapted from Fairley (1977, p. 344).
**Trips in 10 years to Staten Island and Providence

report generated two summary figures. The $p = \frac{1}{4,664,179}$ figure represents the probability that an undesireable LNG event occurs on a single (typical) trip. The number is clearly an average probability over the various trips one could expect over the 10 year horizon time. The Commission interpreted the $\mu = \frac{1}{3746}$ figure as the probability of at least one undesireable LNG event over the 10 year horizon. In fact, this figure is the expected number of such events and only corresponds to the approximate probability of such an event when the number of such events follows a Poisson probability law. No justification is apparent for such an assumption. The risk probability of $\frac{1}{3746}$ was deemed by the Federal Power Commission as low enough so that the proposed LNG tanker and barge movement posed an acceptable risk.

Case Study 2: (Seismic Safety of Nuclear Reactors)

There are approximately 100 light water nuclear reactors currently operational around the United States. One of the environmental hazards potentially leading to a core meltdown is severe seismic activity (earthquakes exceeding 0.2g ground acceleration). In 1974, the U.S. Nuclear Regulatory Commission commissioned a study to quantitatively assess nuclear reactor risks; this included studying the effects of earthquakes. The report generated by that study is known as WASH—1400 or, more informally, the Rasmussen report (see U.S. Nuclear Regulatory Commission (1975)). The NRC concluded that earthquake risk was negligibly small compared to other reactor accident risks. Table 2 summarizes the data and analysis that supported this conclusion.

Three levels of earthquake severity were considered and the resulting overall probability p of core meltdown over a given year

Table 2. Summary of U.S. NRC Report on Probability of Core Melt Due to Earthquake for an Average Site in the Eastern U.S.

Ground acceleration	Earthquake prob. per year	Prob. of damage	Core melt prob. per reactor year
0.2g	7×10^{-4}	3×10^{-5}	2.1×10^{-8}
0.5g	5×10^{-5}	3×10^{-3}	1.5×10^{-7}
1.0g	1×10^{-5}	3×10^{-2}	3×10^{-7}
Overall core melt probability per reactor year			4.7×10^{-7}

at an average (typical) site was computed. The figure arrived at by the Commission was $p = 4.7 \times 10^{-7}$. A project sponsored by the National Science Foundation and under the Directorship of David Okrent (1977, Chapter 9) considered the same problem and arrived at an alternative probability of $p' = 8 \times 10^{-5}$ (assuming minimal or no reactor design errors). This project study differed primarily in its explicit consideration of alternative failure modes leading to core meltdown. This represents a particularly vexing situation for the decision maker who must decide whether or not the risk is acceptable when confronted by conflicting expert opinion. The question arises as how to "average" the two judgments for p. Moreover, how does the decision maker include his personal viewpoint into the reconcilation process?

A number of issues concerning risk assessment are considered in this paper. In Section 2, the general probabilistic structure for evaluating site dependencies is discussed. An information theoretic model is introduced in Section 3; and in Section 4, this model is used for reconciling and updating several expert judgments. A brief discussion of some computational aspects is given in Section 5. Applications of these results to the two Case Studies are considered throughout. Proofs and certain technical material can be found in Appendices A, B, C and D.

2. RISK DETERMINATION WITH SITE DEPENDENCIES

2.1 Relating Risk Determination to Site Occurrences

In this section, we consider a general model applicable for a wide variety of risk determination problems.

Suppose we are interested in assessing the likelihood of an event E which may or may not occur at one or more of M sites. Site is here a generic term that may refer to components within a subsystem, or subsystems within a system, or literally geographical site locations for facilities. It is assumed that E is sufficiently rare that the possibility of two or more occurrences at a single site over the horizon time of interest can be effectively ignored. Let $\Psi_j = 1$, or 0, according to whether, or not, E occurs at site j, so that $\Psi_1, \ldots, \Psi_M$ constitute a complete specification of the outcome of the event process. The event E is regarded as catastrophic were it to occur even once; therefore, our attention is directed to the random variable X denoting the number of occurrences of E. Our interest is in obtaining the probability P_c of a catastrophic event, where $P_c \equiv P(X \geq 1)$. Note that $X = \sum_{j=1}^{M} \Psi_j$. Since X can be expressed in terms of $\Psi_1, \ldots, \Psi_M$, P_c is readily obtainable once the multivariate distribution of $\Psi_1, \ldots, \Psi_M$ is known. If $\Psi_1, \ldots, \Psi_M$ were independent, they would constitute so-called Poisson trials (Feller (1968, p. 218)) and it is not difficult to show that $P_c \geq 1 - e^{-\mu}$, where $\mu = \sum_{j=1}^{M} \pi_j$ is the expected number of occurrences of E over the M sites and $\pi_j \equiv P(\Psi_j = 1)$ is the probability of an occurrence of E at site j, j = 1, ..., M. Moreover, for fixed μ (see Feller (1968, p. 282))

(2.1) $$\lim_{M\to\infty} P_c = 1 - e^{-\mu} .$$

Equation (2.1), therefore, gives P_c in terms of the expected number of occurrences of E, when E is rare, the number of sites is large, and the occurrences are <u>independent</u> across sites.

Unfortunately, the site independence assumption is in most cases not warranted. Commonality of components, facility design, and operational procedures introduces significant dependencies across sites that can not be disregarded. In short, $\Psi_1, \Psi_2, \ldots, \Psi_M$ are typically <u>dependent</u> random variables. (See Appendix A for a technical discussion of this point.) In this case, $\pi_1, \pi_2, \ldots, \pi_M$ do not uniquely define the multivariate distribution of $\Psi_1, \ldots, \Psi_M$. One must in fact specify the probabilities for all possible values that $\Psi_1, \ldots, \Psi_M$ may jointly assume, i.e., for all binary M-vectors. For example, for M = 100 light water reactor sites, the number of distinct probability values required is 2^{100}. Clearly, the task of specifying a multivariate distribution for $\Psi_1, \ldots, \Psi_M$ is impractical in the absence of underlying structural models for the dependencies involved. However, modeling dependency in complex environments such as the ones being considered is currently beyond the state of the art. On the other hand, obtaining reliable judgments concerning each of the marginal distributions of the Ψ_j, or equivalently, of the π_j is certainly reasonable. (Event tree and fault tree analysis are very effective tools in this regard. See Barlow and Proschan (1975, pp. 255-274) for a thorough discussion of these techniques.)

2.2 Risk Probability Bounds and Expert Judgment

Specification of the marginal probabilities $\pi_1, \pi_2, \ldots, \pi_M$ constrains the multivariate probability distribution over $\Psi_1, \Psi_2, \ldots, \Psi_M$ and, therefore, limits the possible multivariate distributions. These constraints in turn give rise to a range of possible values of P_c. If this interval covers the indifference probability, there is uncertainty as to whether or not the potential for catastrophe should be of concern. If, however, the interval lies entirely below the critical value or entirely above the critical value, then we have an unequivocal answer as to where the actual probability of catastrophe P_c lies relative to the threshold probability P_T.

Given $\pi_1, \pi_2, \ldots, \pi_M$, upper and lower bounds for P_c can be derived from the multivariate Fréchet-Hoeffding bounds for multivariate c.d.f.'s (e.g., Dall'Aglio (1972)). (See Appendix A for the derivation.) These bounds are

$$\max_{1 \le i \le M} \pi_i \le P_c \le \min\left(\sum_{i=1}^{M} \pi_i, 1\right). \tag{2.2}$$

For $\sum_{i=1}^{M} \pi_i \le 1$, these bounds are attainable, i.e., there exist actual site dependencies with the noted marginal probabilities giving rise to each P_c in the interval of (2.2). Note that not all of the information in the values $\pi_1, \pi_2, \ldots, \pi_M$ is utilized in constructing these upper and lower bounds. In particular, we need to know only their sum and their largest value.

Now suppose that we wish to construct bounds for P_c, where the only information available concerning the marginal probabilities is that we are given $p = M^{-1} \sum_{i=1}^{M} \pi_i$, which is the site average probability of E occurring at a typical (randomly selected) site. Among the class of all multivariate c.d.f.'s with $\sum_{i=1}^{M} \text{Prob}(\Psi_i = 1) = Mp$, the following tight bounds can be obtained for P_c

(2.3) $$p \leq P_c \leq Mp .$$

Note that the upper bounds of (2.2) and (2.3) are the same while the lower bounds differ. Usually, it is the upper bound on P_c that is of interest, the expectation being that this upper bound will be below the threshold probability; in this situation $\pi_1, \pi_2, \ldots, \pi_M$ provide no additional information beyond that provided by $M^{-1} \sum_{i=1}^{M} \pi_i$.

Equation (2.3) requires a valuation of p, the probability that E occurs at a typical site, and M, the number of sites. This form of expert judgment as model input is natural and quite minimal in its demands on the expert. For example, Lewis notes that the Rasmussen report concerning light water reactor safety focused on "one typical ... reactor." Again note that it is not necessary to know anything about the possible dependencies over the sites to specify p and M, since $p = M^{-1} \sum_{i=1}^{M} \pi_i$ is only a function of the marginal distributions over the sites. It will later be useful for technical reasons to have the expert judgment re-expressed in terms of $\mu = EX$, the mean number of occurrences of E over all sites. Note that

$$\mu = E \sum_{j=1}^{M} \Psi_j = \sum_{j=1}^{M} E\Psi_j = \sum_{j=1}^{M} P(\Psi_j = 1) = \sum_{j=1}^{M} \pi_j = Mp$$

regardless of the multivariate distribution of $\Psi_1, \ldots, \Psi_M$. Hence, (2.3) may be rewritten in terms of μ, without imposing any additional assumptions, as the following

$$M^{-1}\mu \leq P_C \leq \mu \ . \tag{2.4}$$

Accordingly, a specification of the value of μ will formally be referred to hereafter as the expert judgment. Also, because the upper bound in (2.4) is informative only when $\mu < 1$, we will define E as a rare event whenever $\mu < 1$, i.e., whenever expert judgment is that it is "not expected to occur."

Equation (2.4) expresses the range of possible probabilities P_C consistent with a given specification for μ. Of course, there may be considerable uncertainty surrounding the valuation of p and hence μ. It is useful, therefore, to invert (2.4) to find what range of values of μ lead to $P_C \geq P_T$ and $P_C \geq P_T$, respectively. The corresponding lower and upper bounds for μ are given by the following relations

$$\begin{aligned} P_C &\geq P_T \quad \text{if} \quad \mu \geq \bar{\mu}(P_T) \equiv M P_T \\ P_C &\leq P_T \quad \text{if} \quad \mu \leq \underline{\mu}(P_T) \equiv P_T \ . \end{aligned} \tag{2.5}$$

Alternatively, one may ask what range of values of μ is consistent (in the sense of satisfying (2.4)) with a given value for P_C. It follows directly from (2.5) that the interval of consistent values of μ is

$$\underline{\mu}(P_C) \leq \mu \leq \bar{\mu}(P_C) \ . \tag{2.6}$$

The bounds of equation (2.6) may be viewed as providing a sensitivity analysis for P_C in terms of allowable values of μ. In other words, as long as the perceived uncertainty in the true value of μ lies within the bounds given by (2.6), there is no logical inconsistency in adopting the value P_c as the probability of catastrophe.

2.3 Consistency Analysis for LNG Tanker Movements

In a critique of the study concerning the safety of LNG tanker movements, Fairley (1977) notes that the probabilities of most factors could possibly be upwardly corrected. The upward corrections are summarized in the following table taken from Fairley (1974, p. 344).

Table 3. Possible Correction Factors for LNG Tanker Movement Probabilities

Factor Symbol	Source of error or uncertainty in estimate	Possible upward correction factor for estimate (Multiply times original estimated probabilities in Table 1).
B	Reporting error	2 - 5
	Extrapolation to LNG vessels	2 - 5
C	Speculatively based estimate	5
D	Unsubstantiated theory	2+
E	Definitional error	1.5
G	Speculatively based estimate	5

The use of these correction factors (multipliers of 16 and 2 were used for factors B and D, respectively) would upwardly adjust the p from $\frac{1}{4,664,179}$ to $\frac{1}{3,887}$. This, in turn, using the original study methodology would change both μ and P_c from $\frac{1}{3,\text{[illegible]}}$ to $\frac{1}{3.1}$. It appears that this aspect of the critique focuses on possible changes in μ and not on the methodology for obtaining P_c from μ. Based on a consistency analysis (employing (2.6)) for $P_c = \frac{1}{3,746}$, we can see that any μ in the interval $(\frac{1}{3,746}, \frac{1}{3.0})$ could give rise to the noted P_c of $\frac{1}{3,746}$ through use of alternative probability models. Clearly $\mu = \frac{1}{3.1}$ falls in this interval. This analysis then suggests that model selection (or equivalently, as discussed in Section 3, the decision maker's viewpoint) is at least as important as obtaining an expert's judgment that is reliable.

2.4 Risk Determination and the Decision Maker's Viewpoint

As discussed, the various bounds presented will in some cases unequivocally decide the issue of acceptable risk. However, when the range of uncertainty in true P_c given by (2.4) covers the threshold value P_T, we must look to a more detailed model to arrive at a risk determination. The expert judgment, μ, cannot by itself resolve the issue, and it is at this point that the viewpoint of the decision maker must be taken into account. The decision maker retains ultimate responsibility and is not allowed the luxury of being noncommittal. We model his viewpoint about the number of potential occurrences of E by his assessment of the probability distribution governing X. The <u>decision maker's viewpoint</u> will accordingly be represented by a (M+1)-vector $\underset{\sim}{p}$ where p_i equals the decision maker's probability that there will be i occurrences of E. Simple and effective procedures for eliciting such subjective probability distributions exist and are well documented in the literature (see Raiffa (1968)). The viewpoint of the decision maker thus elicited as to the probability distribution of X will not in general be consistent with expert judgment as to the mean of X.

3. AN INFORMATION THEORETIC MODEL

3.1 General Model

The decision maker initially holds to a fixed viewpoint concerning the likelihoods of the number of catastrophes. This viewpoint is expressed as the probability vector $\underset{\sim}{p}$ governing the random variable X. However, given expert judgment that the mean of X is μ, the decision maker is compelled to adjust his viewpoint so as to be consistent with expert judgment. This adjustment process can be expected to result in a new viewpoint as "close" as possible to the initial viewpoint of the decision maker and yet consistent with the expert. There is a natural probabilistic geometry in which this process can be embedded. The expert's judgment is modeled as a linear subspace of the M-dimensional simplex of probability distributions, namely

$$S_\mu = \{(q_0, \ldots, q_M) : \sum_{j=0}^{M} jq_j = \mu, \ \sum_{i=0}^{M} q_i = 1, \text{ and } q_i > 0, \ i = 0, \ldots, M\}. \tag{3.1}$$

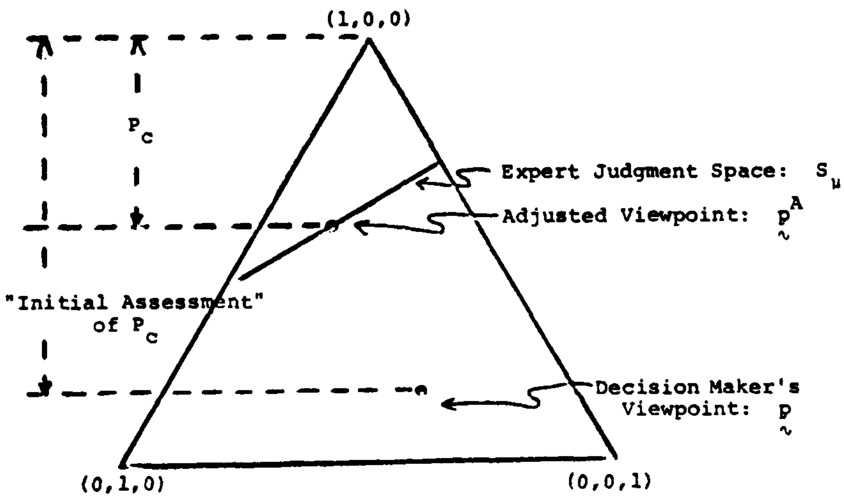
The decision maker's viewpoint modeled as $\underset{\sim}{p} = (p_0, \ldots, p_M)$, where $p_i = \mathrm{Prob}(X = i)$, is also a point in the M-dimensional simplex, but is not in general a point in S_μ. The adjustment process is viewed as finding the adjusted decision maker's viewpoint $\underset{\sim}{p}^\wedge = (p_0^\wedge, \ldots, p_M^\wedge)$, where $\underset{\sim}{p}^\wedge \in S_\mu$ and the "distance" from $\underset{\sim}{p}$ to $\underset{\sim}{p}^\wedge$ is minimized. Note that the probability of a catastrophic event based upon merging the decision maker's viewpoint and the expert judgment is $P_c = 1 - p_0^\wedge$.

Prior to having the expert judgment available, the decision maker's "initial assessment" of P_c was $1 - p_0$.

It is illustrative to examine this in the simple case when M = 2 using baricentric coordinates.

Figure 3.1

Adjusting Viewpoints Based Upon Expert Judgment (M = 2)



Based upon the initial viewpoint of the decision maker, there is an initial value of P_c. By "moving" the decision maker's viewpoint as little as possible and yet conforming with expert judgment, an adjusted viewpoint $\underset{\sim}{p}^A$ is obtained with the corresponding "new" value for P_c.

Within the statistical literature there are a number of possible measures for distances between probability vectors (e.g., Rao (1965, pp. 288-289)). One measure of closeness that has been widely employed is the Kullback-Liebler discriminator $I(\underset{\sim}{a};\underset{\sim}{b})$ between two probability vectors $\underset{\sim}{a} = (a_1, \ldots, a_M)$ and $\underset{\sim}{b} = (b_1, \ldots, b_M)$, where

$$(3.2) \qquad I(a;b) = \sum_{i=0}^{M} a_i \ln(a_i b_i^{-1}) \quad .$$

Formally, therefore, the process of finding the adjusted viewpoint of the decision maker would be equivalent to minimizng $I(q;p)$ over $q \in S_u$ and taking $\hat{p}$ as that (unique) minimizing value of q. The properties of the Kullback-Liebler discriminator are extensively presented in Kullback (1959), and Gokhale and Kullback (1978). Akaike (1977) discusses uses of the Kullback-Leibler discriminator in a number of areas of statistical inference. For a review of other applications, see Sampson and Smith (1979).

There is a considerable literature on this type of adjustment process viewed in the context of generating prior probability distributions for Bayesian inference. An axiomatization of this process together with an extensive discussion of the literature can be found in Shore and Johnson (1980). They demonstrate that the use of any separator other than (3.2) (which they refer to as cross-entropy) for inductive inference when new information is in the form of expected values leads to a violation of one or more reasonable consistency axioms.

This adjustment process has the following interpretation. The conventional Shannon-Wiener (1948) entropy measure of a discrete probability distribution p is given by $H(p) = \sum_{j=0}^{M} p_j \ln p_j^{-1}$. $H(p)$ can be intuitively viewed as the expected "surprise" (see Theil (1967)) associated with p where $\ln p_j^{-1}$ would be the "surprise" generated by the occurrence of event j. Re-writing $I(q,p) = -[\sum_{j=0}^{M} q_j(\ln q_j^{-1} - \ln p_j^{-1})]$, we may interpret $I(q,p)$ as the negative of the expected difference in surprise between viewpoints p versus q if in fact the true probability distribution is $\underset{\sim}{q}$. There is accordingly an inherent asymmetry between $\underset{\sim}{p}$ and $\underset{\sim}{q}$ in the separation measure $I(\underset{\sim}{p},\underset{\sim}{q})$ where $\underset{\sim}{q}$ assumes the role of the "true" probability distribution. Note that this interpretation supports the approach of choosing an adjusted viewpoint from those $\underset{\sim}{q}$ consistent with expert judgment which moreover minimizes $I(\underset{\sim}{q},\underset{\sim}{p})$, i.e., that $\underset{\sim}{q}$ generates the least expected difference in surprise over the initial viewpoint $\underset{\sim}{p}$.

Sampson and Smith have applied this model to the criminalistics problem of assessing the weight of circumstantial evidence linking a suspect with the perpetrator of a crime. A number of results obtained there are also relevant to our problem of assessing P_C. We now paraphrase and summarize these results in the risk determination context; for a formal treatment see Sampson and Smith.

Definition 3.1. A family of distributions $p(i,\tau)$, $-\infty < \tau < \infty$, on the integers 0, ..., M is a finite exponential family of distributions with parameter τ, if for every τ

$$p(i,\tau) = c(\tau)h_i e^{\tau i}, \quad i = 0, \ldots, M,$$

where $h_0 = h_1 = 1$, $h_2 > 0, \ldots, h_M > 0$, and $c(\tau) = \left(\sum_{j=0}^{M} h_j e^{\tau j}\right)^{-1}$. (Choosing $h_0 = h_1 = 1$ basically fixes scale and location origins for the family.)

Result 3.1. (a) Every probability distribution $\underset{\sim}{p} = (p_0, \ldots, p_M)$ on 0, ..., M with $p_i > 0$ belongs to exactly one finite exponential family of distributions.

(b) Within a given finite exponential family of distributions every member distribution is uniquely indexed by the parameter τ and also by $\mu(\tau) \equiv \sum_{i=0}^{M} [i\, c(\tau)h_i e^{\tau i}]$.

Thus every finite exponential family of distributions can be viewed as a one-dimensional curve in the M-dimensional simplex of probability distributions. These curves never intersect and are also space filling in the sense that every point in the interior of the simplex lies on exactly one such curve.

Consider the decision maker's viewpoint $\underset{\sim}{p} = (p_0, \ldots, p_M)$. By Result 3.1 (a), there is a unique finite exponential family $c^*(\tau)h_i^* e^{\tau i}$, $-\infty < \tau < \infty$, such that $p_i = c^*(\tau_0)h_i e^{\tau_0 i}$, $i = 0, \ldots, M$; i.e., $\underset{\sim}{p}$ is that point on the curve indexed by $\tau = \tau_0$.

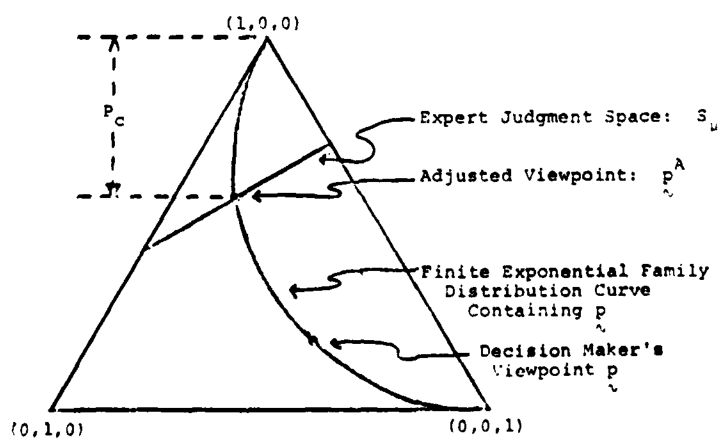
<u>Result 3.2</u>. Based upon the expert judgment that $EX = \mu$, the adjusted viewpoint $\underset{\sim}{p}^A$ can be found by choosing that member of $c^*(\tau)h_i^* e^{\tau i}$, such that $\sum_{i=0}^{M} [i\ c^*(\tau)h_i^* e^{\tau i}] = \mu$.

Intuitively Result 3.2 states that $\underset{\sim}{p}^A$ can be found by tracing the finite exponential family distribution curve containing $\underset{\sim}{p}$ until that curve intersects S_μ. Specifically, the point of intersection is $\underset{\sim}{p}^A$.

Again it is illustrative to examine the probabilistic geometry for the case M = 2, using baricentric coordinates.

<u>Figure 3.2</u>

<u>Finding Adjusted Viewpoints (M = 2)</u>



It is direct to show that (a) $\lim_{\tau\to-\infty} c(\tau)h_i e^{\tau i} = 1$, if $i = 0$, and $= 0$, otherwise; (b) $\lim_{\tau\to\infty} c(\tau)h_i e^{\tau i} = 0$, if $i < M$, and $= 1$, if $i = M$. Thus in terms of Figure 3.2 as τ varies from $-\infty$ to ∞, the corresponding point on the finite exponential family curve traverses from top to lower right bottom. It is visually obvious and analytically easy to show that P_C is monotone increasing in τ. In fact, however, a stronger result may be stated. Let $F(k,\tau) = \sum_{i\le k} c(\tau)h_i e^{\tau i}$ be the c.d.f. of the number of catastrophic events at the M sites. Then in Appendix B, we show that for all k, $F(k,\tau)$ is decreasing in increasing τ. This is equivalent to saying that the probability of more than k catastrophes increases as τ increases and decreases as τ decreases. Thus two different probability vectors on the same finite exponential family curve represent two stochastically ordered views of the likelihoods of catastrophic events, with the viewpoint for a smaller τ corresponding to a belief that the likelihood of k or less catastrophic events is larger than for a viewpoint with a larger τ. Also in Appendix B, we show that the mapping taking the initial decision maker's viewpoint $\underset{\sim}{p}$ to the adjusted viewpoint $\underset{\sim}{p}^A$ is continuous so that small variations in the initial viewpoint lead to small variations in the adjusted viewpoint.

Clearly within the model the determination of P_C is dependent upon the value of μ. However, it is reasonable to expect that as the expert judgment becomes compelling concerning the unlikelihood of a catastrophic event, that is, as $\mu \to 0$, one would find that the initial decision maker's viewpoint becomes increasingly less important in determining P_C. As noted in (2.4), a priori bounds for P_C

are $M^{-1}\mu \leq P_c \leq \mu$. In fact for M = 2 these bounds correspond in Figure 3.2 to the values of P_c determined, respectively, by the points where the "right-hand" and "left-hand" sides of S_μ intersect the boundary of the simplex. Thus, as $\mu \to 0$, these bounds converge to 0, demonstrating $0^{\underline{th}}$ order agreement in P_c based upon different viewpoints. More importantly, Sampson and Smith show that this model provides $1^{\underline{st}}$ order agreement.

Result 3.3. Let $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_2$ be two different viewpoints and denote by $P_c^1(\mu)$ and $P_c^2(\mu)$, the respective probabilities of a catastrophic event based upon expert judgment that $EX = \mu$. Then $P_c^1(\mu) = P_c^2(\mu) + o(\mu) = \mu + o(\mu)$, where $\lim_{\mu \to 0} \frac{o(\mu)}{\mu} = 0$.

3.2 Risk Determination for LNG Tanker Movement

The focus in this case is on obtaining the probability of one or more LNG tanker spills in the New York harbor over a 10 year horizon. The generic term "site" in this context becomes tanker trip, while "catastrophic event" corresponds to an LNG spill. The expert judgment based on the Federal Power Commissions report is that there is an expected total number of LNG spills of $\mu = \frac{1}{3,746}$ with a probability of a spill on any one tanker trip of $\frac{1}{4,664,179}$ (from Table 1). With a total of M = 1245 trips over the 10 year horizon, inequality (2.4) provides upper and lower bounds on the probability of one or more LNG spills P_c as

$$2.4 \times 10^{-7} \leq P_c \leq 2.67 \times 10^{-4} .$$

That is, each of the probabilities in this interval of values is consistent with the FPC report. However, because of the small magnitude of these bounds, the first order approximation provided by Result 3.3 is appropriate in suggesting the value $P_c \approx 2.67 \times 10^{-4}$. Note that this figure is in agreement with that arrived at by the FPC. Moreover, this value represents P_c to $1^{\underline{st}}$ order regardless of the initial viewpoint. Thus, two decision makers with differing points of view when presented with $\mu = \frac{1}{3,746}$ would agree based upon the model that the given figure for P_c is correct to the $1^{\underline{st}}$ order.

In order to illustrate the actual adjustment process, suppose the decision maker's viewpoint can be modeled as $\underset{\sim}{p} = (.99, .009, .0009, .00009, .00001, 0., \ldots, 0.)$. Thus the decision maker initially believes that the probability of catastrophe is .01. Based upon expert judgment that $\mu = \frac{1}{3,746} = .0002669$, the adjusted viewpoint is $\underset{\sim}{p}^A = (.999742, .26 \times 10^{-3}, .75 \times 10^{-6}, .22 \times 10^{-8}, .70 \times 10^{-11}, 0., \ldots, 0.)$. Hence, based upon the given FPC judgment for μ and the decision maker's hypothetical viewpoint $\underset{\sim}{p}$, the probability of one or more LNG spills in 1245 trips is $1. - .999742 = 2.574 \times 10^{-4}$.

4. MULTIPLE EXPERT JUDGMENTS

4.1 Updating and Merging

Up to this point, we have considered the case where there is just one expert judgment available. Now suppose that two judgments are available to the decision maker. There appears to be two basic situations in this case. One is where both experts are coequal and their judgments are to be equally and simultaneously assimilated by the decision maker in the process of adjusting his initial viewpoint.

The other basic situation is where the decision maker adjusts his viewpoint in light of a first expert's judgment and later is confronted with an updated expert judgment to which he must readjust accordingly. The latter situation is considered first.

Sampson and Smith noted that there are two ways to readjust the decision maker's viewpoint in light of a superseding expert judgment. One approach is to discard the viewpoint adjusted to the first expert's judgment and return to the initial viewpoint; then the decision maker adjusts in the usual fashion to the second expert judgment. The other approach is for the decision maker to act as if the viewpoint adjusted to the first expert judgment is the new initial decision maker's viewpoint; then the decision maker would adjust the new viewpoint to take into account the second and superseding expert judgment. Sampson and Smith show that based upon the model, both of these two noted approaches are equivalent.

Now suppose that two coequal experts provide judgments μ_1 and μ_2 where it is assumed that $\mu_1 < \mu_2$. Based on his viewpoint, the decision maker must simultaneously merge both experts' judgments in obtaining the adjusted viewpoint.

Denote the decision maker's viewpoint by $\underset{\sim}{p}$ and let $\underset{\sim}{p}_1^A$ and $\underset{\sim}{p}_2^A$ be the two possible adjusted viewpoints corresponding to μ_1 and μ_2, respectively. Let $\underset{\sim}{p}^*$ denote the decision maker's adjusted viewpoint where the decision maker in arriving at that viewpoint must merge the two experts' judgments μ_1 and μ_2. Within the context of the model and under the assumption of coequal experts, we require

$$I(\underset{\sim}{p}^*;\underset{\sim}{p}_1^A) = I(\underset{\sim}{p}^*;\underset{\sim}{p}_2^A),$$

that is, the decision maker adopts an adjusted viewpoint which is equally separated from the viewpoints that would be taken based on each expert separately. Roughly speaking, $\underset{\sim}{p}^*$ is "informationally equidistant" from $\hat{\underset{\sim}{p}}_1$ and $\hat{\underset{\sim}{p}}_2$.

Writing $\underset{\sim}{p}$ in its finite exponential family form $f(i,\tau_0)$, where $f(i,\tau) = c(\tau)h_i e^{\tau i}$, we can by Results 3.1 and 3.2 represent $\hat{\underset{\sim}{p}}_j$ by $f(i,\tau_j)$, $j = 1, 2$, where $\tau_1 < \tau_2$. Let $\mu(\tau) = \sum_{i=0}^{M} i f(i,\tau)$. Without any further constraints on $\underset{\sim}{p}^*$, it is shown in Lemma C.1 (Appendix C) that any $\underset{\sim}{p}^* = (p_0^*, \ldots, p_M^*)$ satisfying

$$(4.1) \qquad \sum_{i=0}^{M} i\, p_i^* = E(\mu(T)),$$

where T is a random variable with uniform distribution on (τ_1,τ_2), is "informationally equidistant" from $\hat{\underset{\sim}{p}}_1$ and $\hat{\underset{\sim}{p}}_2$. Another way of viewing (4.1) is that $E(\mu(T))$ is the merged value of μ_1 and μ_2 as seen by the decision maker with viewpoint $\underset{\sim}{p}$. Note that the merged value of μ_1 and μ_2 is dependent upon the specific viewpoint of the decision maker.

However, within the context of the model, it is required that $\underset{\sim}{p}^*$ be as informationally close as possible to $\underset{\sim}{p}$. Specifically, we require that $\underset{\sim}{p}^*$ minimize $I(\underset{\sim}{p}^*;\underset{\sim}{p})$ subject to $I(\underset{\sim}{p}^*;\hat{\underset{\sim}{p}}_1) = I(\underset{\sim}{p}^*;\hat{\underset{\sim}{p}}_2)$. In Lemma C.2 (Appendix C), it is shown that $\underset{\sim}{p}^*$ is given by $f(i,\tau^*)$, where τ^* is determined uniquely by

$$(4.2) \qquad \mu(\tau^*) = E(\mu(T)),$$

where T has a uniform distribution on (τ_1,τ_2). A graphical presentation of this process employing baricentric coordinates for the M = 2 case is given in Figure 4.1; and in Figure 4.2, we view the process for a general distribution graphing $\mu(\tau)$ versus τ.

Figure 4.1

Merging Two Coequal Expert Judgments, Baricentric Representation (M = 2)

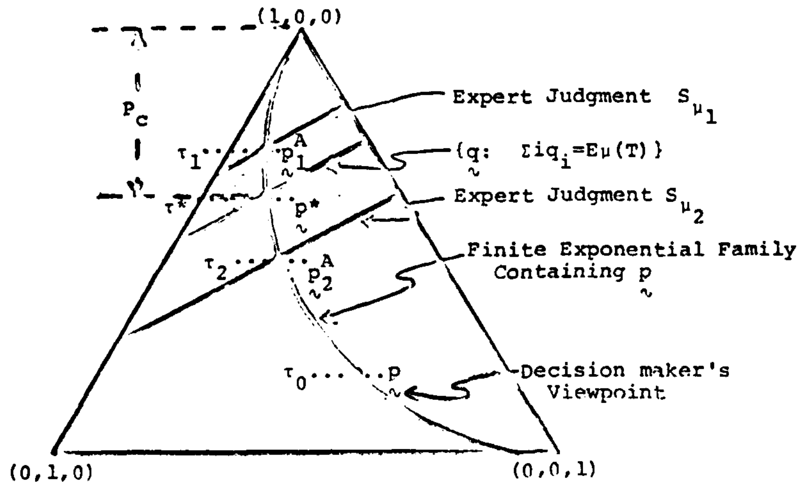
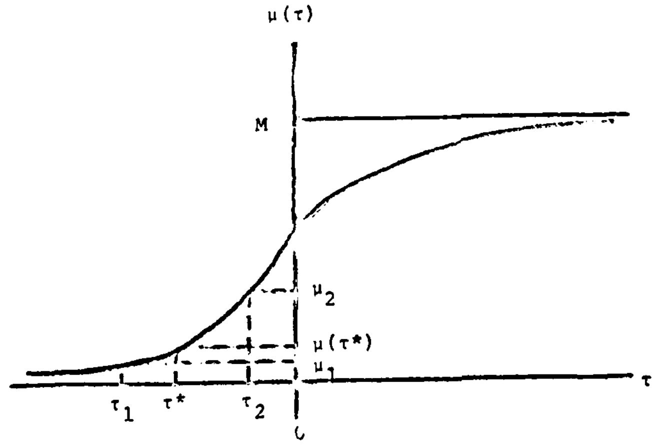


Figure 4.2

Merging Two Coequal Expert Judgments For a Given Viewpoint Family



Intuitively, the result of (4.2) says that the way to merge μ_1 and μ_2 is to average the intermediate means by weighting them uniformly with respect to τ along the finite exponential family curve containing $\underset{\sim}{p}$. Clearly this is different than weighting uniformly with respect to the means themselves; such an approach would yield $(\mu_1 + \mu_2)/2$ as a merged opinion. More specifically, from Figure 4.2 and (4.2), we are able to see that τ^* is chosen so that the shaded area under the $\mu(\tau)$ function is equal in area to a rectangle whose sides are of length $\tau_2 - \tau_1$ and $\mu(\tau^*)$. Moreover, depending on where τ_1 and τ_2 are located relative to the curvature of $\mu(\tau)$, $\mu(\tau^*)$ may be closer to μ_1 or to μ_2. In the case where τ_2 is "moderate" and τ_1 is "small," $\mu(\tau^*)$ tends to be closer to μ_1 than to μ_2.

The actual evaluation requires first obtaining τ_1 and τ_2 corresponding to $\underset{\sim}{p}_1^A$ and $\underset{\sim}{p}_2^A$, respectively. (Several techniques for this are discussed in Section 5.) Then it can be shown (see Appendix C) that

$$\mu(\tau^*) = -(\tau_2-\tau_1)^{-1}\int_{\tau_1}^{\tau_2}\left(\frac{\sum_{i=0}^{M} i\,(p_0/p_1)^i p_i e^{si}}{\sum_{i=0}^{M} (p_0/p_1)^i p_i e^{si}}\right)ds,$$

where $\underset{\sim}{p} = (p_0, p_1, \ldots, p_M)$ is the decision maker's viewpoint.

4.2 Binomial Viewpoints and Rare Expert Judgments

Two special cases for merging are worth considering further. One is where the decision maker has a binomial viewpoint and the other is where μ_1 and μ_2 are small. Suppose that for some π, p_i is given by $\binom{M}{i}\pi^i(1-\pi)^{M-i}$, $i = 0, \ldots, M$, and the two experts' judgments are μ_1 and μ_2. In Appendix C, it is shown that μ^*, the merged judgment from the decision maker's viewpoint, can be explicitly stated in terms of μ_1, μ_2, and M, as follows:

$$\mu^* = M \ \ell n[\frac{M - \mu_1}{M - \mu_2}](\ell n[\frac{\mu_2}{\mu_1} \cdot \frac{M - \mu_1}{M - \mu_2}])^{-1}.$$

In the second interesting case, we suppose that the expert judgments μ_1 and μ_2 are quite small. For this case, it can be shown (see Appendix C) that

$$(4.3) \qquad \mu^* = (\mu_2 - \mu_1)/(\ell n\ \mu_2 - \ell n\ \mu_1) + o(\max(\mu_1,\mu_2)).$$

Incidentally, if the decision maker were allowed to have a Poisson viewpoint, the exact value for μ^* would be $(\mu_2 - \mu_1)/(\ell n \mu_2 - \ell n \mu_1)$. Note that if instead we are dealing with $p^* = \mu^*/M$, $p^{(1)} = \mu_1/M$, and $p^{(2)} = \mu_2/M$, then (4.3) can be re-expressed as

$$(4.4) \qquad p^* \approx (p^{(2)} - p^{(1)})/(\ell n\ p^{(2)} - \ell n\ p^{(1)}) \ .$$

Now suppose $\mu_1 = 10^{-v_1}$ and $\mu_2 = 10^{-v_2}$, where v_1 is substantially larger than v_2; then from (4.3), it follows that μ^* can be approximated by $10^{-(v_2+\log_{10}v_1)}$.

4.3 Applications of Merging Judgments

The result of (4.4) can be used to merge rare probability judgments concerning SCRAM failures in nuclear reactors. The SCRAM system in boiling-water reactors is a system whereby the nuclear chain reaction in a reactor can be stopped quickly. In the Rasmussen report, the probability of a SCRAM failure was assessed. Lewis writes

"The problem encountered in estimating this probability is that the SCRAM systems are so important and so well designed that the event in question (SCRAM failure on demand) has never happened and therefore there is no basis in experience from which to assess the reliability of this system."

Based on two different sets of assumptions, the Rasmussen report presented two distinctly different probabilities that a typical SCRAM operation fails, namely $p^{(1)} = 10^{-12}$ and $p^{(2)} = 10^{-6}$. In an attempt to combine these rare judgment probabilities, the geometric average of 10^{-9} was employed by the Rasmussen report. This procedure for combining these judgments prompted Lewis to comment "there may somewhere be a statistician who believes that this is a valid procedure but he has yet to make himself known." An application of (4.4) to $p^{(1)} = 10^{-12}$ and $p^{(2)} = 10^{-6}$ suggests that the combined p^* should be 1.67×10^{-7} which is a couple of orders of magnitude less conservative than the Rasmussen report value of 10^{-9}. Note that in the case of rare expert judgments, the value of p^* essentially does not depend on the decision maker's viewpoint.

Another application of (4.3) is to merging judgments concerning the seismic safety of nuclear reactors in Case Study 2. In this case, P_c becomes the probability of one or more reactor meltdowns during a given year. The sites are literally light water nuclear reactors. The expert judgment in this case flows from two distinct sources. The NRC report placed the mean number of meltdowns over the M = 100 reactor sites at a value of $\mu_1 = 4.7 \times 10^{-5}$. On the other hand, the Okrent study assessed the figure at $\mu_2 = 8 \times 10^{-3}$. Noting that these figures are sufficiently small so as to justify use of a first order approximation, we can merge the two expert judgments according to (4.3), obtaining $\mu^* = 1.55 \times 10^{-3}$. Note that the effect of the adjustment process is to give greater weight to the more conservative figure corresponding to the judgment of the Okrent study. The ultimate probability figure for P_c will depend on the initial viewpoint of the decision maker. Again, however, to first order, the merged value for the probability of one or more reactor meltdowns due to severe seismic activity is $P_c \approx 1.55 \times 10^{-3}$.

5. COMPUTATIONAL CONSIDERATIONS

Brockett, Charnes and Cooper (1978) have in essence shown that the actual derivation of $\underset{\sim}{p}^A$ from $\underset{\sim}{p}$ and μ can be viewed as an unconstrained convex programming problem. Consequently, standard computer packages can be employed. Other mathematical techniques that can be used in the constrained Kullback-Leibler discrimination minimization problem can be found in Kullback and Gokhale. In Appendix D, we give the details for an IMSL (1979) implementation of the Brockett et al technique.

Another approach that could be used to finding $\underset{\sim}{p}^A$ is described in Lemma D.1 (Appendix D). Let m(s) be the moment generating function determined by the decision maker's viewpoint probabilities, i.e., $m(s) \equiv \sum_{i=0}^{M} e^{si} p_i$. Let s_0 be the unique solution to

$$m'(s) - \mu m(s) = 0 ;$$

then the i^{th} entry of $\underset{\sim}{p}^A$ is given by

$$p_0^{-1} p_i (p_0/p_1)^i e^{\tau_0 i} \left(\sum_{j=0}^{M} p_0^{-1} p_j (p_0/p_1)^j e^{\tau_0 j} \right),$$

where $\tau_0 = s_0 - \ell n(p_0/p_1)$.

6. DISCUSSION AND SUMMARY

In this paper we have presented an information theoretic model that provides a minimum bias approach for the merging of expert judgment and decision maker's opinion as to the likelihood of rare, but catastrophic events. In particular, the decision maker expresses his viewpoint concerning the probabilities of various numbers of catastrophic event occurrences across a number of sites, and the expert provides his judgment concerning the likelihood of a catastrophic event occuring at a typical site. When the expert's judgment is that the likelihood of a catastrophe at a typical site is remote, we show that decision makers, with different viewpoints, would still agree (to 1st order) concerning the probability of catastrophe.

Using the basic model, we have developed meaningful and objective methods to merge and to update judgments from more than one expert. Approximations are given for merging of two expert judgments when the judgments indicate the probabilities of catastrophes are very small. The results for this rare event merging are independent of the decision maker's initial viewpoint.

Several computational approaches suitable for computer implementation are discussed.

Lewis wrote concerning "the importance to society of assessing risk in quantitative terms and of making sound interpretations of those risks." We believe that our modelling approach provides an important step toward attaining this important societal goal.

Appendix A

SITE DEPENDENCIES AND PROBABILITY BOUNDS

A.1 Dependencies Across Sites

We discuss here the relationship between dependencies among random variables versus independent random variables sharing an unknown but common parameter. Parameter here may represent design commonality, component commonality or operating system commonality.

In particular consider the random variables $\Psi_1, \Psi_2, \ldots, \Psi_M$ serving as indicators for occurrence or non-occurrence of catastrophic events over the M sites. Suppose all failures at these sites share a common component type and manufacturer. Let q represent the probability that the design of that component type by that manufacturer is defective. Then $q \equiv P(Y = 1)$, where $Y = 1$, or 0, respectively, if the design is defective, or not. Now suppose

$$P(\Psi_i = 1 | Y = y) = \begin{cases} \pi_i & \text{for } y = 1 \\ \pi_i' & \text{for } y = 0, \end{cases}$$

where $\pi_i \neq \pi_i'$, for some i. An expert reliably testifies as to the likelihood p that a randomly selected site would experience a malfunction over the horizon time. Then

$$p = p(\Psi_I = 1 | Y = 1) P(Y = 1) + P(\Psi_I = 1 | Y = 0) P(Y = 0)$$

$$= qM^{-1} \sum_{i=1}^{M} \pi_i + (1-q) M^{-1} \sum_{i=1}^{M} \pi_i'$$

where I is the index of a randomly selected site.

Lemma A.1†: The random variables $\Psi_1, \Psi_2, \ldots, \Psi_M$ are conditionally

† We assume here for simplicity that the component of interest represents the only commonality among the sites.

independent given Y.

Proof: Immediate. ∎

Lemma A.2: The random variables $\Psi_1, \Psi_2, \ldots, \Psi_M$ are unconditionally dependent.

Proof: Suppose $\pi_1 \neq \pi_1'$. Then

$$P(\Psi_1 = 1 | \Psi_2 = 1) = \frac{\pi_1 \pi_2 q + \pi_1' \pi_2' (1 - q)}{\pi_2 q + \pi_2' (1 - q)}$$

$$\neq \pi_1 q + \pi_1'(1 - q) = p(\Psi_1 = 1).$$ ∎

Since we are not in general aware of the value of Y, i.e., whether or not a design error has been made, we are in the situation of Lemma A.2.

The above analysis can be easily generalized to the case of multiple common components or subsystems by letting Y be a binary vector of dimension equal to the number of distinct shared components. Obviously in practice, the specification of a complex system in this fashion is essentially impossible.

A.2 Bounds for P_C

A proof of the upper and lower bounds of (2.2) for the general case of $\Psi_1, \Psi_2, \ldots, \Psi_M'$ following an arbitrary multivariate distribution follows.

Theorem A.1: Suppose $\Psi_1, \Psi_2, \ldots, \Psi_M$ are zero-one random variables with $\pi_i = \text{Prob}(\Psi_i = 1)$ being given. Let $X = \sum_{i=1}^{M} \Psi_i$, and $P_C = p(X \geq 1)$. Then

$$\text{(A.1)} \qquad \max_{1 \leq i \leq M} \pi_i \leq P_C \leq \min\left(\sum_{i=1}^{M} \pi_i, 1\right).$$

Moreover, these bounds are attainable, whenever $EX \leq 1$.

Proof: Let $F_{\Psi}(t)$ denote the joint c.d.f. of $\Psi_1, \ldots, \Psi_M$, where the i^{th} marginal c.d.f. is $F_i(t_i) = 0$, for $t_i < 0$; $= p_i$, for $0 \leq t_i \leq 1$; and $= 1$, otherwise. Then for all $\underset{\sim}{t}$

(A.2) $$\max(0, \sum_{i=1}^{M} F_i(t_i) - (M - 1)) \leq F_{\underset{\sim}{\Psi}}(\underset{\sim}{t}) \leq \min_{1 \leq i \leq M} \{F_i(t_i)\}.$$

(See Dall'Aglio.) Note that $P_c = 1 - F_{\underset{\sim}{\Psi}}(\underset{\sim}{0})$, so that (A.2) immediately implies (A.1). The attainment of the lower bound of (A.1) follows from the fact that $\min_{1 \leq i \leq M} \{F_i(t_i)\}$ is a c.d.f. If $\sum_{i=1}^{M} \pi_i \leq 1$, the upper bound of (A.1) is attainable because the lower bound of (A.2) is itself a c.d.f. (see Dall'Aglio or Conway (Theorem 5.2; 1979)). $\square$

Another way of viewing Theorem A.1 is to define $\Phi_1 = \{F_{\underset{\sim}{\Psi}}(\underset{\sim}{t})$: for all i, $P(\Psi_i = 1) = \pi_i$ and $P(\Psi_i = 0) = 1 - \pi_i\}$. Then thinking of P_c as a function of $F_{\underset{\sim}{\Psi}}(\underset{\sim}{t}) \in \Phi_1$, (A.1) provides attainable upper and lower bounds on P_c as $F_{\underset{\sim}{\Psi}}(\underset{\sim}{t})$ varies over Φ_1. Now suppose that full marginal information is not known and all the information that is available is $M^{-1} \sum_{i=1}^{M} \pi_i = p$. Define $\Phi_2 = \{F_{\underset{\sim}{\Psi}}(\underset{\sim}{t})$: for all i, $P(\Psi_i = 0) + P(\Psi_i = 1) = 1$, and $(M^{-1}) \sum_{i=1}^{M} P(\Psi_i = 1) = p\}$. Clearly $\Phi_1 \subsetneq \Phi_2$. Then as $F_{\underset{\sim}{\Psi}}(\underset{\sim}{t})$ ranges over Φ_2, obtainable upper and lower bounds for P_c are given by (2.3).

Appendix B

STOCHASTIC ORDERING AND CONTINUITY OF ADJUSTMENT PROCESS

B.1 Stochastic Ordering of Exponential Family Distributions

Definition B.1 (E.g., Lehmann (p. 73, 1959)). A family of cumulative distribution functions $\{F_\tau\}$ is stochastically increasing (decreasing) in τ if $\tau < \tau'$ implies $F_\tau(x) \geq (\leq)\ F_{\tau'}(x)$ for all x.

It follows from Lehmann (1959; Corollary 2, p. 70 and Lemma 2, p. 74) that $F_\tau(x) = \sum_{i=0}^{[x]} c(\tau)h_i e^{\tau i}$, $x \leq M$, is stochastically increasing in τ. A more direct proof follows below, but first we require a simple probability lemma of interest in itself.

Lemma B.1. Let X be a random variable taking on value i with probability ω_i, where i = 0, ..., M. Then for all k,

$$(EX)(P(X \leq k)) \geq \sum_{i=0}^{k} i\omega_i \ . \qquad \text{(B.1)}$$

Proof: For any $k \leq [EX]$, where $[\cdot]$ denotes the greatest integer function, we have $\sum_{i=0}^{k} (i - EX)\omega_i \leq 0$ so that (B.1) is immediate. Suppose that there exists $k_o > [EX]$ such that $\sum_{i=0}^{k_o} (i - EX)\omega_i > 0$. However, because

$$0 = \sum_{i=0}^{M} (i - EX)\omega_i = \sum_{i=0}^{k_o} (i - EX)\omega_i + \sum_{i=k_o+1}^{M} (i - E(X))\omega_i ,$$

we obtain an immediate contradiction concerning the existence of such a k_o. $\Box$

Theorem B.1. Let $p(i,\tau) = c(\tau)h_i e^{\tau i}$, i = 0, ..., M and let $F(k,\tau) = \sum_{i=0}^{k} p(i,\tau)$. Then $F(k,\tau)$ is a decreasing function in τ for all k = 0, ..., M.

Proof: Note that since $\mu(\tau) = -c'(\tau)/c(\tau)$,

$$\frac{dF(k,\tau)}{d\tau} = \sum_{i=0}^{k} c'(\tau)h_i e^{\tau i} + \sum_{i=0}^{k} ic(\tau)h_i e^{\tau i}$$

$$= -[(EX)(P(X \le k)) - \sum_{i=0}^{k} iP(X = i)],$$

where X is a random variable with p.d.f. $p(i,\tau)$. The result is now immediate from Lemma B.1. $\sqcap$

B.2 Sensitivity of the Adjusted Viewpoint to Variation in the Initial Viewpoint

Theorem B.2. For every $0 \le \mu \le M$, the function taking $\underset{\sim}{p}$ into $\underset{\sim}{p}^{\Lambda}$ is continuous.

Proof: Let $\{\underset{\sim}{p}_k\}$ be a sequence of viewpoints converging to the viewpoint $\underset{\sim}{p}$. We need to show that $\lim_{k\to\infty} \underset{\sim}{p}_k^{\Lambda} = \underset{\sim}{p}^{\Lambda}$. Let $\underset{\sim}{h} = (h_0, \ldots, h_M)$ define (in the sense of Definition 3.1) the finite exponential family for $\underset{\sim}{p}$ and $\underset{\sim}{p}^{\Lambda}$, and let $\tau(\underset{\sim}{h})$ and $\tau^{\Lambda}(\underset{\sim}{h})$ be the unique respective parameters. Similarly define $\underset{\sim}{h}_k$, $\tau(\underset{\sim}{h}_k)$, and $\tau^{\Lambda}(\underset{\sim}{h}_k)$. Then it follows from Sampson and Smith (1979, Equation (4.2)) that $\lim_{k\to\infty} \underset{\sim}{h}_k = \underset{\sim}{h}$. Note that $g(\tau^{\Lambda}(\underset{\sim}{h}_k), \underset{\sim}{h}_k) = 0$ for all k where we define for fixed μ

$$g(\tau,h) = \sum_{j=1}^{m} jh_j e^{\tau j} \left(\sum_{j=0}^{m} h_j e^{\tau j}\right)^{-1} - \mu.$$

Because $\frac{\partial g}{\partial \tau} = \sigma_\tau^2 > 0$ (see Sampson and Smith (1979, Equation (4.3)) and $\frac{\partial g}{\partial \underset{\sim}{h}_k}$ is continuous, it follows that $\tau^{\Lambda}(\underset{\sim}{h})$ is a continuous function of $\underset{\sim}{h}$ by the implicit function theorem (e.g., Bartle (1964; p. 260)). Hence, $\lim_{k\to\infty} \tau^{\Lambda}(\underset{\sim}{h}_k) = \tau^{\Lambda}(\underset{\sim}{h})$. Denote the i^{th} entry of a vector $\underset{\sim}{x}$ by $x(i)$. It follows from the preceding that

$$\lim_{k\to\infty} p_k^A(i) = \lim_{k\to\infty} h_k(i)e^{\tau^A(h_k(i))} \left(\sum_{j=0}^{M} h_k(j)e^{\tau^A(h_k(j))}\right)^{-1}$$

$$= h(i)e^{\tau^A(h(i))} \left(\sum_{j=0}^{M} h(j)e^{\tau^A(h(j))}\right)^{-1}$$

$$= p^A(i), \text{ for all } i = 0, \ldots, M. \qquad \sqcap$$

Appendix C

MERGING COEQUAL EXPERT JUDGMENTS

C.1 The General Case

Lemma C.1 describes which viewpoints are "informationally equidistant" from two adjusted viewpoints determined by a decision maker from two coequal expert judgments. Lemma C.2 utilizes Lemma C.1 to choose among these viewpoints that viewpoint "closest" to the decision maker's viewpoint.

Lemma C.1. Suppose that $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_2$ are given, respectively, by $f(i,\tau_1)$ and $f(i,\tau_2)$, where $\tau_1 < \tau_2$ and $f(i,\tau) = c(\tau)h_i e^{\tau i}$, $i = 0, \ldots, M$. Let $\mu(\tau) = \sum_{i=0}^{M} if(i,\tau)$. Then a necessary and sufficient condition that $\underset{\sim}{p}^* = (p_0^*, \ldots, p_M^*)$ satisfy $I(\underset{\sim}{p}^*;\underset{\sim}{p}_1) = I(\underset{\sim}{p}^*;\underset{\sim}{p}_2)$ is that $\sum_{i=0}^{M} ip_i^* = E(\mu(T))$, where T is a random variable having uniform distribution on (τ_1, τ_2).

Proof: If $I(\underset{\sim}{p}^*;\underset{\sim}{p}_1) = I(\underset{\sim}{p}^*;\underset{\sim}{p}_2)$, then

$$\text{(C.1)} \qquad \sum_{i=0}^{M} p_i^* \ \ln\left(\frac{p_i^*}{p_{1i}}\right) = \sum_{i=0}^{M} p_i^* \ \ln\left(\frac{p_i^*}{p_{2i}}\right) ,$$

where p_{1i}, p_{2i}, and p_i^* are, respectively the (i+1)st entries of $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$, and $\underset{\sim}{p}^*$. Equation (C.1) and the form of $f(i,\tau)$ imply

$$\sum_{i=0}^{M} p_i^*[\ln c(\tau_1) + \ln h_i + \tau_1 i] = \sum_{i=0}^{M} p_i^*[\ln c(\tau_2) + \ln h_i + \tau_2 i],$$

which, in turn, implies

$$\text{(C.2)} \qquad \sum_{i=0}^{M} i\ p_i^* = -(\tau_2-\tau_1)^{-1}(\ln c(\tau_2) - \ln c(\tau_1)) .$$

By differentiating with respect to τ the equation

$$\sum_{i=0}^{M} c(\tau) h_i e^{\tau i} = 1,$$

we obtain

(C.3) $\quad \mu(\tau) = -(dc(\tau)/d\tau)/c(\tau),$

and, hence, from (C.3), we may represent

(C.4) $\quad \ln c(\tau) = -\int_{-\infty}^{\tau} \mu(s)ds + K,$

where K is a constant. Substituting (C.4) into (C.2), we obtain

(C.5) $\quad \sum_{i=0}^{M} i\, p_i^* = (\tau_2-\tau_1)^{-1} \int_{\tau_1}^{\tau_2} \mu(s)ds,$

and the result now follows. $\sqcap$

Lemma C.2. Suppose that $\underset{\sim}{p}$, $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_2$ are given, respectively, by $f(i,\tau_0)$, $f(i,\tau_1)$, and $f(i,\tau_2)$, where $\tau_1 < \tau_2$ and $f(i,\tau) = c(\tau)h_i e^{\tau i}$, $i = 0, \ldots, M$. Let $\mu(\tau) = \sum_{i=0}^{M} i\, f(i,\tau)$. Then

$$\min_{\underset{\sim}{q}:I(\underset{\sim}{q};\underset{\sim}{p}_1)=I(\underset{\sim}{q};\underset{\sim}{p}_2)} I(\underset{\sim}{q};\underset{\sim}{p})$$

occurs uniquely at $\underset{\sim}{q} = \underset{\sim}{p}^*$, where $\underset{\sim}{p}^* = f(i,\tau^*)$ and τ^* is determined by $\mu(\tau^*) = E(\mu(T))$, where T is a random variable having a uniform distribution on (τ_1,τ_2).

Proof: This follows immediately from Lemma C.1 and Result 3.2. $\sqcap$

A more specific representation for $\mu(\tau^*)$ in terms of the decision maker's viewpoint can be obtained. Let $\underset{\sim}{p} = (p_0, \ldots, p_M)$ be the viewpoint with $p_0 > 0$ and $p_1 > 0$. Sampson and Smith (1979, (4.2)) showed that the finite exponential family containing $\underset{\sim}{p}$ can be parametrized by

(C.6) $$f(i,\tau) = c(\tau)p_0^{-1}p_i(p_0/p_1)^i e^{\tau i} ,$$

where $c(\tau)$ is chosen so that $\sum_{i=0}^{M} f(i,\tau) = 1$. When $\tau = \ell n(p_1/p_0)$, $f(i,\tau) = p_i$. Again noting that $\mu(\tau) = -c'(\tau)/c(\tau)$, and using (C.6), we can rewrite (C.5) for computational purposes as

$$\mu(\tau^*) = -(\tau_2-\tau_1)^{-1}\int_{\tau_1}^{\tau_2}\left(\frac{\sum_{i=0}^{M} i(p_0/p_1)^i p_i e^{si}}{\sum_{i=0}^{M} (p_0/p_1)^i p_i e^{si}}\right)ds.$$

C.2 The Binomial Case

Now suppose that the decision maker has a binomial viewpoint, i.e.,

(C.7) $$p_i = \binom{M}{i}\pi^i(1 - \pi)^{M-i}.$$

Then (C.7) can be parametrized as

$$f(i,\tau) = e^{M\tau}\binom{M}{i}e^{i\tau},$$

where $\tau = \ell n(\pi/(1-\pi))$. If we parametrize by $\mu = M\pi$, then $\tau = \ell n(\mu/(M - \mu))$. Suppose that we have two expert opinions $\mu_1 < \mu_2$, with corresponding $\tau_1 < \tau_2$. By Lemma C.1, the merged opinion μ^* is given by

$$\mu^* = (\tau_2 - \tau_1)^{-1} \int_{\tau_1}^{\tau_2} \frac{Me^s}{1 + e^s}\, ds$$

$$= M(\tau_2 - \tau_1)^{-1}\ell n[(1 + e^{\tau_2})/(1 + e^{\tau_1})].$$

Substituting in terms of μ_1, μ_2, we have

$$\mu^* = M\ \ell n\left[\frac{M - \mu_1}{M - \mu_2}\right]\left(\ell n\left[\frac{\mu_2}{\mu_1} \cdot \frac{M - \mu_1}{M - \mu_2}\right]\right)^{-1} .$$

C.3 Merging Rare Judgments

We now consider rare judgment approximations; that is, the case where μ_1 and μ_2 are small. Sampson and Smith (1979, Equation (5.2) show that

$$\mu(\tau) = e^{\tau} + \varepsilon(\tau), \tag{C.8}$$

where $\lim_{\tau\to-\infty} \varepsilon(\tau)/\mu(\tau) = 0$.

Let $\tau_1 < \tau_2$ correspond to expert judgments $\mu_1 < \mu_2$. Then by (C.5) and (C.8),

$$\mu^* = \alpha + \beta,$$

where $\alpha = (\tau_2-\tau_1)^{-1}(e^{\tau_2} - e^{\tau_1})$ and $\beta = (\tau_2-\tau_1)^{-1}\int_{\tau_1}^{\tau_2}\varepsilon(s)ds$.

Observe that $\varepsilon(s) = \mu(s) - e^{s}$ is the difference of two continuous functions and hence is also continuous. Therefore, $\varepsilon(s)$ attains its maximum value over $[\tau_1,\tau_2]$. Let $\varepsilon(\tau_{12}^*) = \max_{\tau_1\le\tau\le\tau_2} \varepsilon(\tau)$. Then

$$\beta \le (\tau_2-\tau_1)^{-1}\varepsilon(\tau_{12}^*)(\tau_2-\tau_1) = \varepsilon(\tau_{12}^*).$$

It follows that

$$\beta = o(\mu_2),$$

because as $\tau_2 \to -\infty$

$$\frac{\beta}{\mu_2} = \frac{\beta}{e^{\tau_2}}\,\frac{e^{\tau_2}}{\mu_2} \le \frac{\varepsilon(\tau_{12}^*)}{e^{\tau_{12}^*}}\,\frac{e^{\tau_2}}{\mu_2} \to 0, \text{ as } \mu_2 \to 0.$$

Now, by (C.8), $\mu(\tau)/e^{\tau} \to 1$ as $\tau \to -\infty$, and since $\tau(\mu) = \mu^{-1}(\mu)$ (see Sampson and Smith (1979, Equation (4.3))), we obtain that

$\mu/e^{\tau(\mu)} \to 1$ as $\mu \to 0$, which implies that $\tau(\mu) = \ln \mu + o(\mu)$. Hence,

$$\alpha = (\ln \mu_2 + o(\mu_2) - \ln \mu_1 - o(\mu_1))^{-1}(e^{o(\mu_2)}\mu_2 - e^{o(\mu_1)}\mu_1).$$

Note that

$$\frac{1}{\mu_2}\left[\frac{\mu_2 - \mu_1}{\ln \mu_2 - \ln \mu_1} - \alpha\right] = \frac{o(\mu_2)}{\mu_2}\left[\frac{\mu_2 - \mu_1}{(\ln\mu_2 - \ln\mu_1)^2 + o(\mu_2)(\ln\mu_2 - \ln\mu_1)}\right]$$

$$\to 0, \text{ as } \mu_2 \to 0,$$

so that it now follows that

$$\mu^* = \frac{\mu_2 - \mu_1}{\ln \mu_2 - \ln \mu_1} + o(\mu_2) \ .$$

Appendix D

COMPUTING $\underset{\sim}{p}^A$ BASED UPON μ AND $\underset{\sim}{p}$

Brockett, et al have shown that the problem of minimizing the Kullback-Liebler discrimination subject to constraints on the probability vector obtained can be re-expressed as an unconstrained convex mathematical program. Specializing their results to our case results in the following mathematical program:

$$\text{(P)} \qquad \underset{z_1, z_2}{\text{Minimize}}\; y = e^{z_2 - 1}\left(\sum_{i=0}^{m} p_i e^{iz_1}\right) - \mu z_1 - z_2 \ .$$

Letting the solution to (P) be z_1^*, z_2^*, we obtain

$$p_i^A = p_i e^{iz_1^* + z_2^* - 1} \qquad\qquad i = 0, \ldots, M.$$

The program (P) may be quickly and easily solved by any unconstrained mathematical programming algorithm. For example, IMSL (International Mathematical and Statistical Library) has a FORTRAN callable subroutine ZXMIN which will solve (P) after defining the function y as a user supplied FORTRAN subroutine. For problems where a large number of p_i are positive or μ is very small additional care must be taken to avoid computational numerical difficulties.

Another approach to computing p^A makes use of certain relationships between finite exponential families containing a viewpoint $\underset{\sim}{p}$ and the moment generating function corresponding to $\underset{\sim}{p}$. Through this approach, the optimization problem becomes one of finding the unique root of a specific equation. One of the benefits of this approach is that for small M, the problem can be solved reasonably readily on a programable hand calculator.

Lemma D.1. Suppose $\underset{\sim}{p} = (p_0, \ldots, p_M)$, $p_0 > 0$, $p_1 > 0$, is the decision maker's viewpoint and let $f(i,\tau)$, given by (C.6) be the finite exponential family containing $\underset{\sim}{p}$. Then the unique τ_0 which satisfies $\mu(\tau_0) = \mu_0$ is $\tau_0 = s_0 - \ln(p_0/p_1)$, where s_0 is the solution to

(C.7) $$m'(s) - \mu_0 m(s) = 0,$$

where $m(s)$ is the moment generating function of $\underset{\sim}{p}$.

Proof: The equation $\mu(\tau) = \mu_0$ can be rewritten as

(C.8) $$p_0^{-1} \sum_{i=0}^{M} i\ p_i (p_0/p_1)^i e^{\tau i} = \mu_0\ c(\tau)^{-1}$$

$$= \mu_0 p_0^{-1} \sum_{i=0}^{m} p_i (p_0/p_1)^i e^{\tau i}\ .$$

Since $m(\tau) = \sum_{i=0}^{m} p_i e^{\tau i}$, (C.8) can be rewritten as

$$m'(\tau + \ln(p_0/p_1)) = \mu_0 m(\tau + \ln(p_0/p_1))$$

and the result then follows. $\square$

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